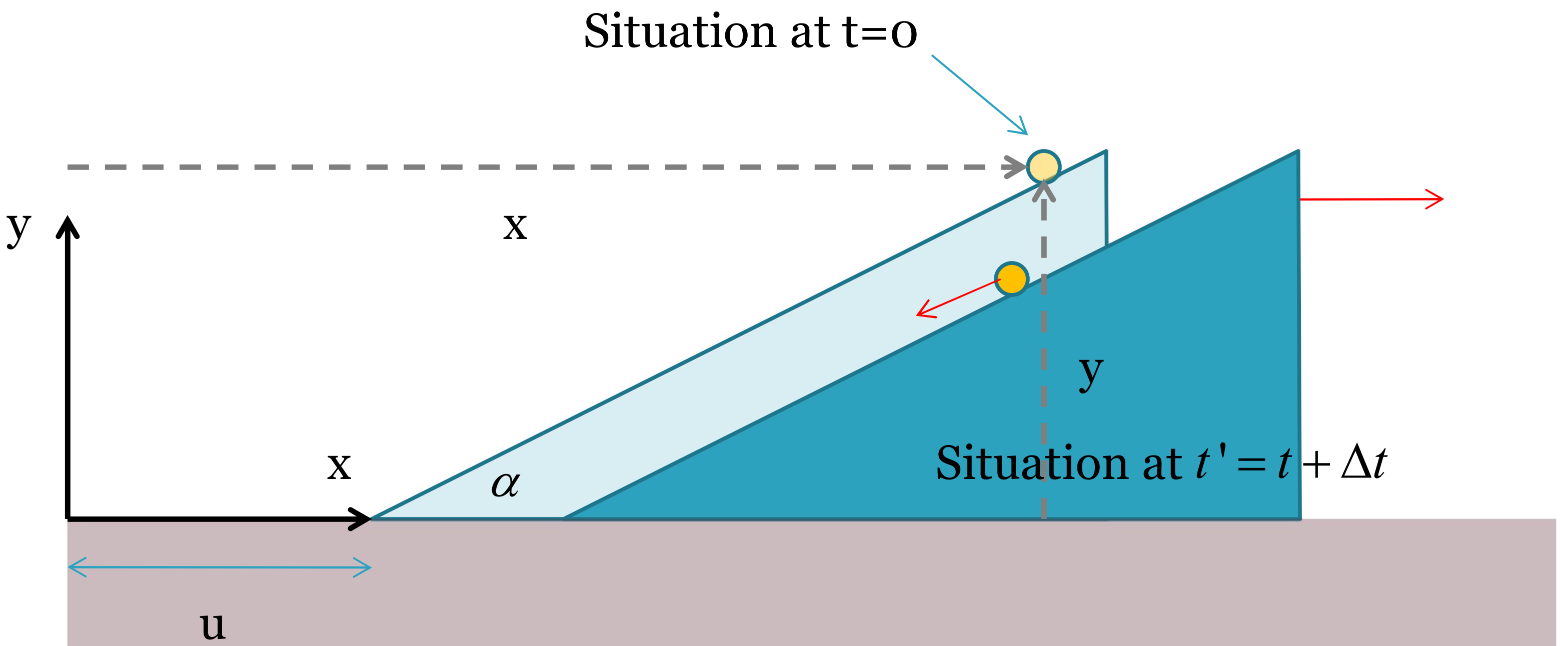
PHYS 705: Classical Mechanics

-HW 5 : bead sliding down an incline

Regular Cartesian Coordinates:
-for the bead (x,y fixed on the ground)
-for the wedge (just one u)

Constraint: $\tan\alpha = \dfrac{y}{(x-u)}$



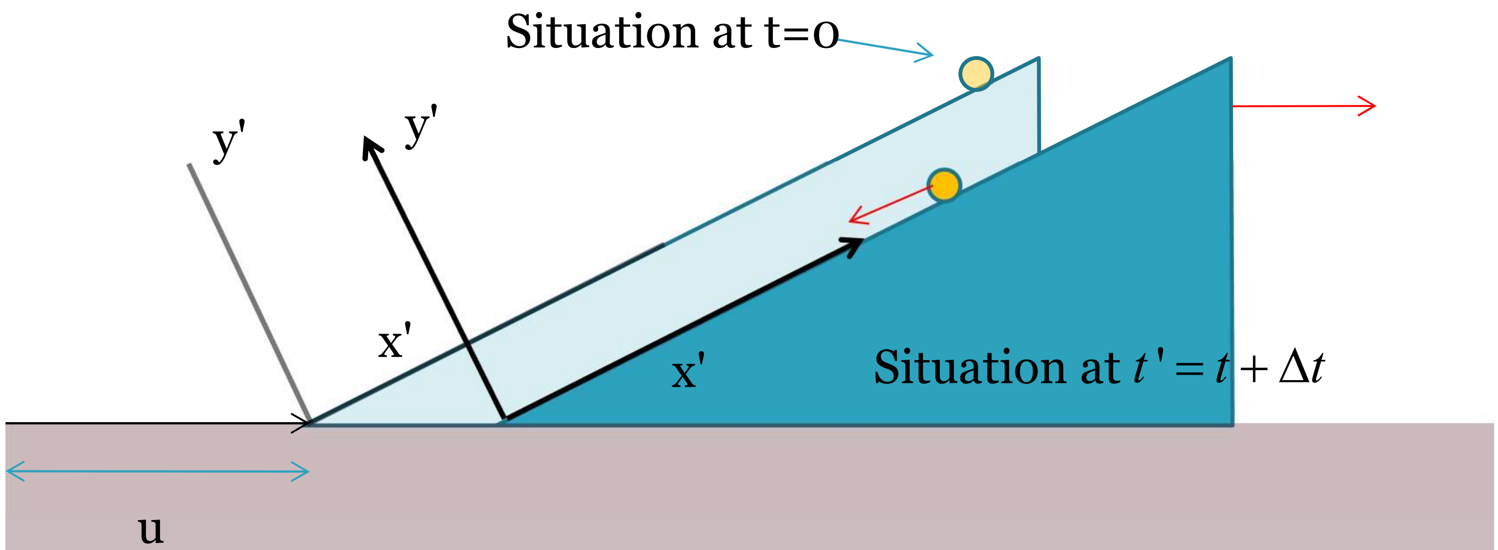
The position of the wedge is simple to specify. It is simply given by u.

A better set of generalized coordinates:

-one for the bead (fixed on the incline)

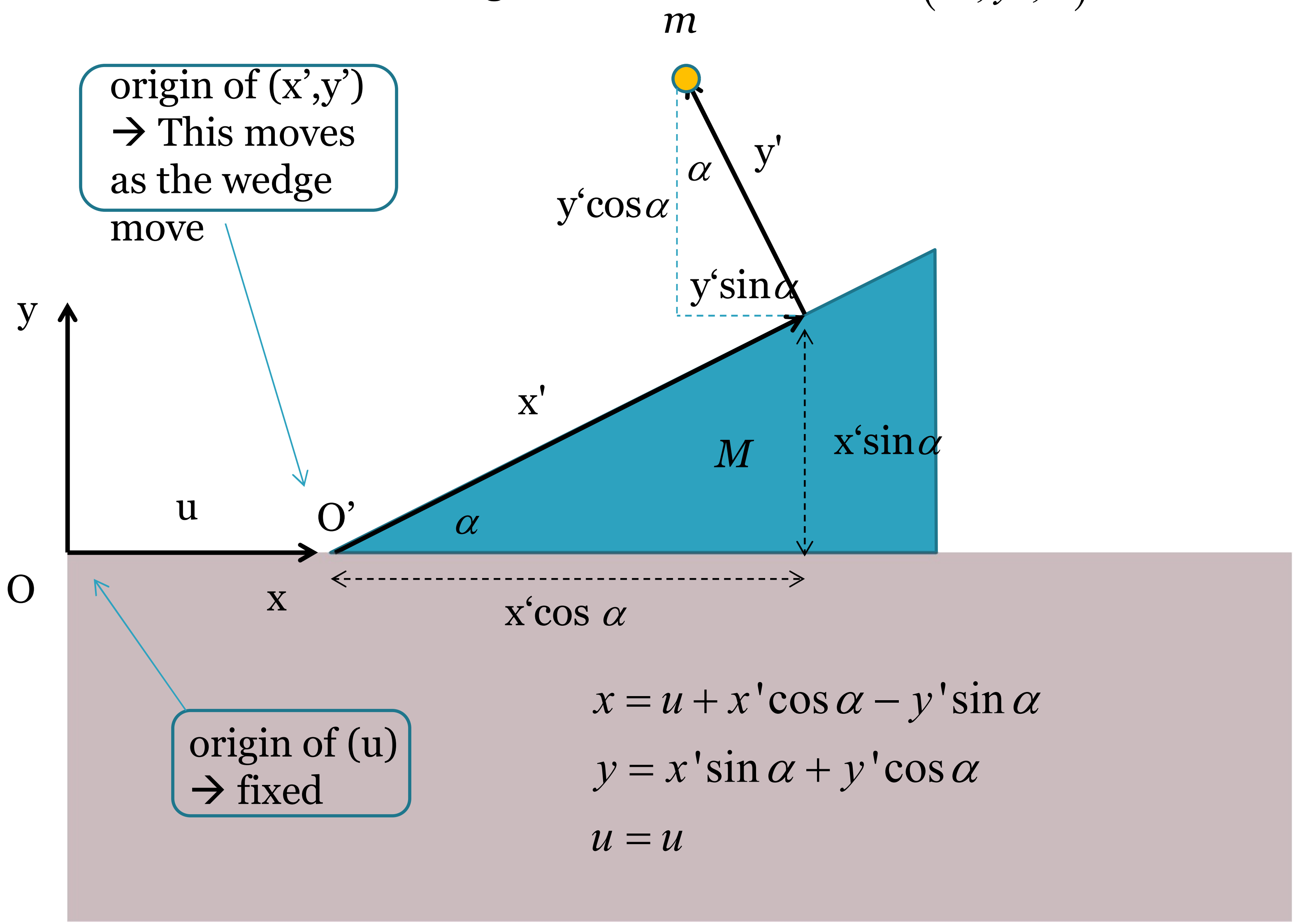
-one for the wedge (fixed on the ground)

Constraint: $y' = 0$



The position of the wedge is simple and it is simply given by u. And, you now need to write out the expressions in describing the position of the bead using these x', y', and u.

Linking the Cartesian coordinates (x, y, u)
and the slanted generalized coordinates: (x', y', u)

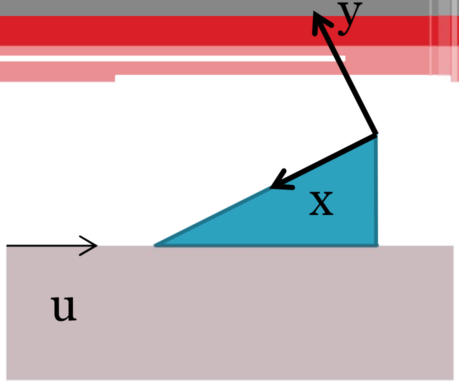


$$x = u + x'\cos\alpha - y'\sin\alpha$$

$$y = x'\sin\alpha + y'\cos\alpha$$

$$u = u$$

System Set-up



$$T = \frac{1}{2}m(\dot{x}_m^2 + \dot{y}_m^2) + \frac{1}{2}M\dot{u}^2$$
$$= \frac{1}{2}(M+m)\dot{u}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + 2\dot{u}\dot{x}\cos\alpha - 2\dot{u}\dot{y}\sin\alpha)$$

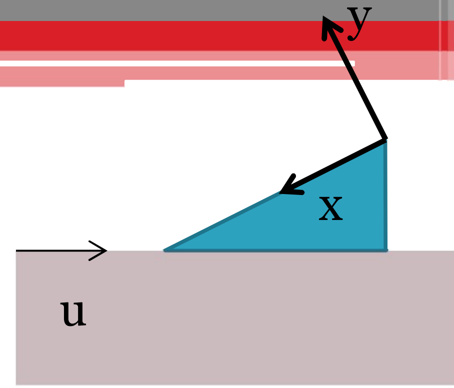
$$V = mg\left(x\sin\alpha + y\cos\alpha\right)$$

$$L = T - V$$

$$g(x, y, u) = y = 0$$

L is slightly more complicated but g is simple !

EOM and Constraint Force



$$\ddot{x} = -\frac{(M+m)g\sin\alpha}{M+m\sin^2\alpha}$$

$$\ddot{y} = 0$$

$$\ddot{u} = \frac{mg\sin\alpha\cos\alpha}{M+m\sin^2\alpha}$$

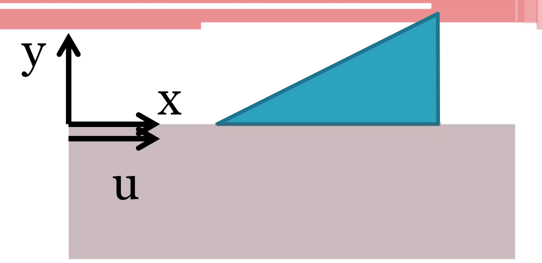
$$\lambda = \frac{Mg\cos\alpha}{M+m\sin^2\alpha}$$

Constraint force is also very simple here.

$$Q_y = \lambda\frac{\partial g}{\partial y} = \lambda = \frac{Mg\cos\alpha}{M+m\sin^2\alpha}$$

$$W_{\text{contstraint}} = Q_y\Delta y = \frac{Mg\cos\alpha}{M+m\sin^2\alpha}\cdot 0 = 0 \qquad \text{since } \Delta y = 0$$

System Set-up



$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} M\dot{u}^2$$

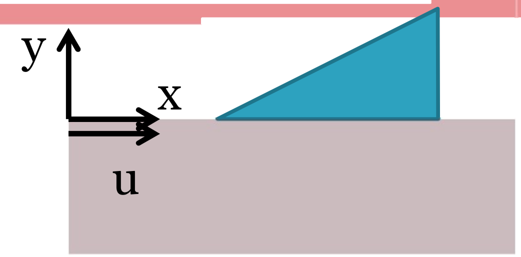
$$V = mgy$$

$$L = T - V$$

$$g(x, y, u) = y - (x - u)\tan\alpha = 0$$

L is simple but g is slightly more complicated !

EOM and Constraint Force



$$\ddot{x} = -\frac{\lambda}{m}\tan\alpha$$

$$\ddot{y} - (\ddot{x} - \ddot{u})\tan\alpha = 0$$

$$\ddot{u} = \frac{\lambda}{M}\tan\alpha$$

$$\lambda = \frac{mg}{1 + \left(1 + \frac{m}{M}\right)\tan^2\alpha}$$

Constraint force is more complicated here.

$$Q_x = -\lambda\tan\alpha, \quad Q_y = \lambda, \quad Q_u = \lambda\tan\alpha,$$

$$W_{\text{contstraint}} = Q_x\Delta x + Q_y\Delta y + Q_u\Delta u = 0 \qquad \text{but } \Delta x \neq 0, \Delta y \neq 0, \Delta u \neq 0,$$

Focus-Directrix Formulation: Summary

Summary on conic sections for the Kepler orbits:

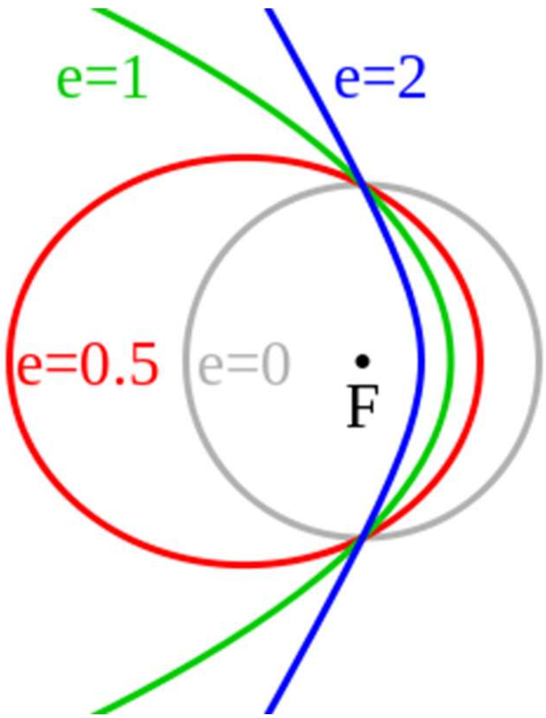
$$r = \frac{\alpha}{1+\varepsilon\cos\theta} \qquad \alpha = \frac{l^2}{mk} \qquad \varepsilon = \sqrt{1+\frac{2El^2}{mk^2}}$$

$$\begin{array}{lll} \varepsilon > 1 & E > 0 & \textit{hyperbola} \\ \varepsilon = 1 & E = 0 & \textit{parabola} \\ 0 < \varepsilon < 1 & -\frac{mk^2}{2l^2} < E < 0 & \textit{ellipse} \\ \varepsilon = 0 & E = -\frac{mk^2}{2l^2} & \textit{circle} \end{array}$$

(ellipse, circle): $E = -\frac{k}{2a}$

Focus-Directrix Formulation: Summary

Kepler orbits with different energies $(E \leftrightarrow \varepsilon)$ but with the same angular momentum $(l \leftrightarrow \alpha)$



Motion in Time: True & Eccentric Anomaly

Goal: To simplify the integration for the EOM

$$t = \int_{r_0}^{r} dr \Bigg/ \sqrt{\frac{2}{m}\left(E + \frac{k}{r} - \frac{l^2}{2mr^2}\right)}$$

$\psi \equiv$ *eccentric anomaly*

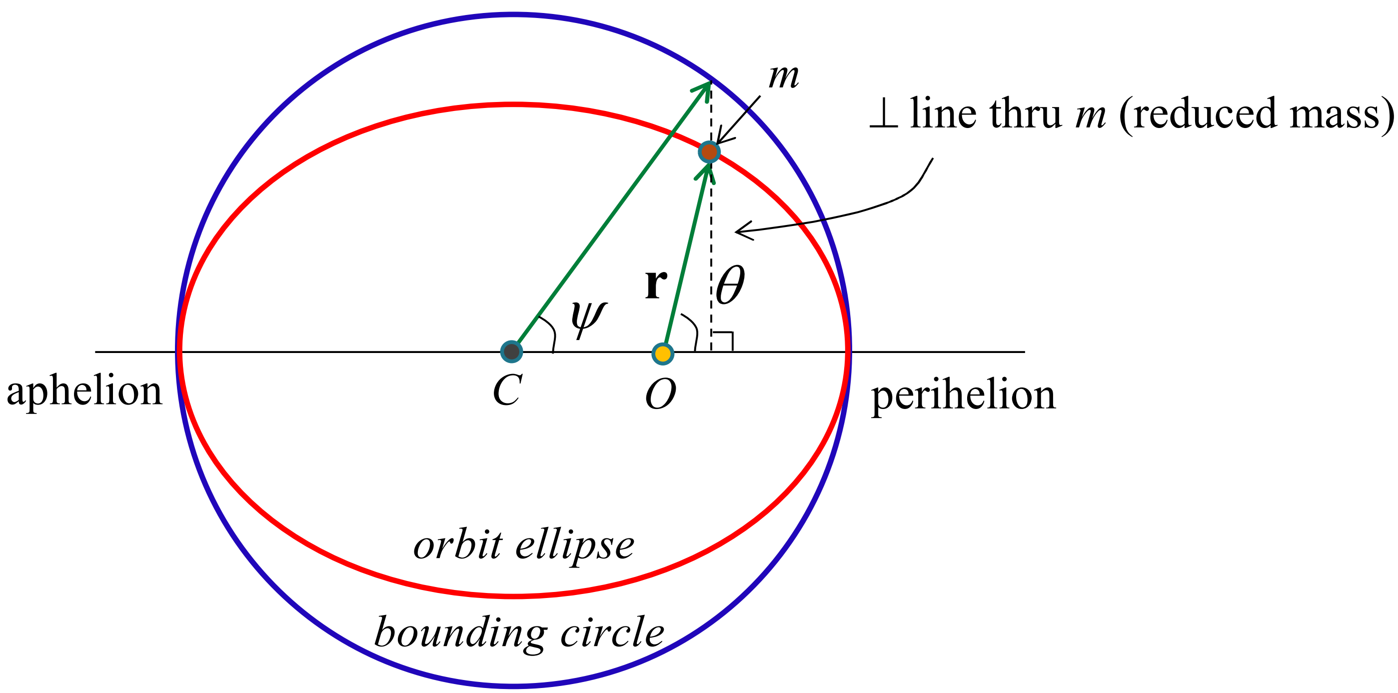
defined by: $r = a\left(1 - \varepsilon \cos\psi\right)$ (basically, a change of variable $r \to \psi$)

Historically, θ is called the true anomaly and it can be shown that

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\left(\frac{\psi}{2}\right)$$

True & Eccentric Anomaly: Geometry

The true and eccentric anomaly are related geometry as follows:



(O is at one of the focii of the orbit ellipse)

(C is the center of bounding circle)

Motion in Time: Eccentric Anomaly

Defining the mean anomaly M as: $M = \dfrac{2\pi}{\tau} t$

→ We arrive at the **Kepler's Equation**:

$$\boxed{M = \psi - \varepsilon \sin \psi}$$

Standard (non-numeric) procedure in solving celestial orbit equation:

1. Solve Kepler's equation → $\psi(t)$ [transcendental]
2. Use the $\psi \leftrightarrow \theta$ transformation to get back to the true anomaly $\theta(t)$.

Legendre Transform

Here is the definition of the Legendre Transform for $F(x)$

$$G(s) = sx(s) - F(x(s))$$

Note that $G(s)$ is a function of s and we have to express $F(x(s))$ in terms of s by inverting the relation: $s = \dfrac{dF(x)}{dx}$ to get $x(s)$

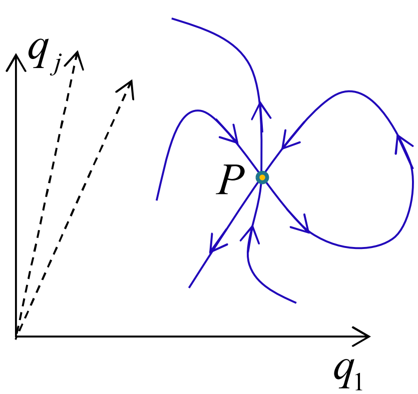
Configuration Space vs. Phase Space

① A given point in configuration space $\left(q_1,\cdots,q_n\right)$ prescribes fully the "configuration" of the system at a given time *t*.

→ However, the specification of a point in this space does NOT specify the *time evolution* of the system completely !

(a unique soln for a *n*-dim 2^{nd} – order ODE needs 2*n* ICs)

→ Many different paths can go thru a given point in config space



← Different paths crossing *P* will have the same set of $\left\{q_j\right\}_1^n$ but diff $\left\{\dot{q}_j\right\}_1^n$

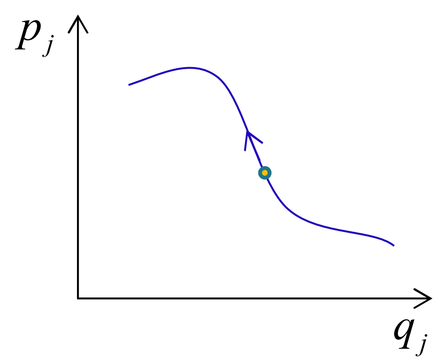
Configuration Space vs. Phase Space

② To specify the state AND time evolution of a system uniquely at a given time, one needs to specify BOTH $\{q_j\}$ AND $\{\dot{q}_j\}$ or equivalently,

$$\{q_j, p_j\}$$

→ The 2n-dim space where both $\{q_j\}$ and $\{p_j\}$ are independent variables is called *phase space*.

→ Thru any given point in phase space, there can only be ONE unique path !



GOAL: to find the EOM that applies to points in phase space.